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Graded reflection equations and the one-dimensional small-polaron open chain

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Abstract. The graded reflection equations are presented to describe quantum integrable lattice fermion open chains on a finite interval with independent boundary conditions on each end. Specifically, the general boundary K supermatrices are determined for the one-dimensional small-polaron open chain. The result is consistent with the system under study admitting Lax pair formulation.

In recent years, there has been considerable interest in the study of completely integrable quantum lattice spin open chains [1–8]. As was shown by Sklyanin [1], there is a variant of the usual formalism of the quantum inverse scattering method (QISM) [9–11], which may be used to describe systems on a finite interval with independent boundary conditions on each end. Central to his approach is the introduction of an algebraic structure called the reflection equations (RE) [12]. Although Sklyanin's argument was carried out only for the P- and T-invariant R matrices, it is now known that the formalism may be extended to apply to any systems integrable by the quantum R-matrix approach [8]. Recently, the boundary K-matrices have been constructed by several groups [6] for the Heisenberg spin-$\frac{1}{2}$ open chain and by the present author [7, 8] for the one-dimensional (1D) coupled spin open chains related with the 1D Hubbard open chain and the 1D Bariev open chain.

However, many integrable periodic chains of interacting fermions relevant to low-dimensional condensed matter physics are known in the literature [13–16]. A common feature of these chains is that their integrability is described by the graded Yang–Baxter equation [17]. Thus it seems interesting to establish the graded version of Sklyanin's formalism to treat quantum integrable fermion open chains; this paper is addressed to this question.

Specifically, the graded reflection equations are presented for the 1D small-polaron open chain. From this the general boundary K-supermatrices are solved. Our result is consistent with the system admitting the Lax pair formulation [18].

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Let us start from the 1D small-polaron open chain with Hamiltonian [18]

$$H = -\sum_{j=2}^{N}[(a_j^\dagger a_{j-1} + a_{j-1}^\dagger a_j) + \tfrac{1}{2}\cos(2\eta)(2n_j - 1)(2n_{j-1} - 1)] \\ + \sin(2\eta)\cot\xi_+ n_N + \alpha_+ a_N^\dagger + \beta_+ a_N + \sin(2\eta)\cot\xi_- n_1 + \alpha_- a_1^\dagger + \beta_- a_1 \,. \tag{1}$$

Here $a_j^\dagger$ and a_j are, respectively, the creation and annhilation operators at lattice site j, and satisfy the usual anticommutation relations

$$\{a_j, a_k\} = \{a_j^\dagger, a_k^\dagger\} = 0 \qquad \{a_j, a_k^\dagger\} = \delta_{jk} \tag{2}$$

and n_j is the density operator, $n_j = a_j^\dagger a_j$. Furthermore, η is a coupling parameter and $\xi_\pm, \alpha_\pm$ and $\beta_\pm$ are some members of the Grassmann algebra with $\xi_\pm$ even and $\alpha_\pm, \beta_\pm$ odd satisfying $\alpha_\pm\beta_\pm = 0$.

For our purposes let us now recall some basic results for the 1D small-polaron chain with periodic boundary conditions [13]. In [13] it was shown that the Hamiltonian of the 1D small-polaron periodic chain commutes with the transfer matrix, which is the supertrace of the monodromy matrix $T(u)$,

$$T(u) = L_N(u) \cdots L_1(u) \tag{3}$$

where

$$L_j(u) = \begin{pmatrix} (\mathrm{i}\sin(u+2\eta) - \sin u)n_j + \sin u & \sin 2\eta a_j \\ -\mathrm{i}\sin 2\eta a_j^\dagger & (-\mathrm{i}\sin u - \sin(u+2\eta))n_j + \sin(u+2\eta) \end{pmatrix}. \tag{4}$$

Here u is the spectral parameter and the local monodromy matrix $L_j(u)$ acts in the supertensor product of the physical (Hilbert) space W_j and the auxiliary superspace V, and the supertrace as well as the supermatrix products are carried out in the auxiliary superspace V. Moreover, it is also shown that the elements of the supermatrix $T(u)$ are the generators of an associative superalgebra T defined by the relations

$$R_{12}(u_1 - u_2)\overset{1}{T}(u_1)\overset{2}{T}(u_2) = \overset{2}{T}(u_2)\overset{1}{T}(u_1)R_{12}(u_1 - u_2) \tag{5}$$

where

$$R_{12}(u_1 - u_2) = \\ \begin{pmatrix} \sin(u_1 - u_2 + 2\eta) & 0 & 0 & 0 \\ 0 & -\mathrm{i}\sin(u_1 - u_2) & \sin 2\eta & 0 \\ 0 & \sin 2\eta & \mathrm{i}\sin(u_1 - u_2) & 0 \\ 0 & 0 & 0 & -\sin(u_1 - u_2 + 2\eta) \end{pmatrix} \tag{6}$$

and

$$\overset{1}{X} \equiv X \otimes \mathrm{id}_{V_2} \qquad \overset{2}{X} \equiv \mathrm{id}_{V_1} \otimes X$$

for any supermatrix $X \in \mathrm{End}(V)$. For later use, we list some useful properties enjoyed by the R-matrix:

(i) unitarity: $R_{12}(u)R_{21}(-u) = \rho(u)$

(ii) crossing-symmetry: $R_{12}^{st_2}(-u - 4\eta)R_{21}^{st_1}(u) = \tilde{\rho}(u)$

where $\rho(u)$ and $\tilde{\rho}(u)$ are some scalar functions:

$$\rho(u) = -\sin(u + 2\eta)\sin(u - 2\eta) \qquad \tilde{\rho}(u) = -\sin u \sin(u + 4\eta)\,. \tag{7}$$

In order to describe integrable lattice fermion chains with the boundary conditions different from the periodic ones, we introduce two associative superalgebras $\mathcal{T}_+$ and $\mathcal{T}_-$ defined by the given R-matrix $R(u_1 - u_2)$ and the relations

$$R_{12}(u_1 - u_2) \overset{1}{\mathcal{T}}_- (u_1) R_{21}(u_1 + u_2) \overset{2}{\mathcal{T}}_- (u_2) = \overset{2}{\mathcal{T}}_- (u_2) R_{12}(u_1 + u_2) \overset{1}{\mathcal{T}}_- (u_1) R_{21}(u_1 - u_2) \tag{8}$$

and

$$\begin{aligned} R_{21}^{st_1 \mathrm{i}st_2}(-u_1 + u_2) \overset{1}{\mathcal{T}}{}_+^{st_1} (u_1) R_{12}(-u_1 - u_2 - 4\eta) \overset{2}{\mathcal{T}}{}_+^{\mathrm{i}st_2} (u_2) &= \overset{2}{\mathcal{T}}{}_+^{\mathrm{i}st_2} (u_2) R_{21}(-u_1 - u_2 - 4\eta) \\ \times \overset{1}{\mathcal{T}}{}_+^{st_1} (u_1) R_{12}^{st_1 \mathrm{i}st_2}(-u_1 + u_2) \end{aligned} \tag{9}$$

respectively. Here the supertransposition $st_\alpha (\alpha = 1, 2)$ is only carried out in the αth factor superspace of $V \otimes V$, whereas ist_α denotes the inverse operation of st_α. By modification of Sklyanin's arguments [1], one may show that the quantities $t(u)$ given by

$$\tau(u) = \mathrm{str}(\mathcal{T}_+(u)\mathcal{T}_-(u))$$

constitute a commutative family, i.e.

$$[\tau(u_1), \tau(u_2)] = 0 \,. \tag{10}$$

A class of important realizations of the superalgebras $\mathcal{T}_+$ and $\mathcal{T}_-$ consists in choosing $\mathcal{T}_\pm(u)$ in the form

$$\mathcal{T}_-(u) = T_-(u)\tilde{\mathcal{T}}_-(u)T_-^{-1}(-u) \tag{11}$$

and

$$\mathcal{T}_+^{st}(u) = T_+^{st}(u)\tilde{\mathcal{T}}_+^{st}(u)T_+^{a}(-u) \tag{12}$$

with

$$T_-(u) = L_M(u) \cdots L_1(u) \tag{13}$$

$$T_+(u) = L_N(u) \cdots L_{M+1}(u) \tag{14}$$

and

$$\tilde{\mathcal{T}}_\pm(u) = K_\pm(u) \tag{15}$$

where $K_\pm(u)$ are representations of $\mathcal{T}_\pm$ in the Grassmann algebra.

We now solve (8) and (9) for $K_+(u)$ and $K_-(u)$. For the R-matrix (6), one may check that the supermatrix $K_-(u)$ given by

$$K_-(u) = -\frac{1}{\sin\xi_-} \begin{pmatrix} \sin(u - \xi_-) & \frac{\alpha_- \sin\xi_- \sin 2u}{\mathrm{i}\sin 2\eta} \\ \frac{\beta_- \sin\xi_- \sin 2u}{\mathrm{i}\sin 2\eta} & -\sin(u + \xi_-) \end{pmatrix} \tag{16}$$

satisfies (8), whereas the supermatrix $K_+(u)$ can be obtained from the isomorphism of the algebras $\mathcal{T}_-$ and $\mathcal{T}_+$. Indeed, given a solution $\mathcal{T}_-$ of (8), the quantities $\mathcal{T}_+(u)$ defined by

$$\mathcal{T}_+^{st}(u) = \sigma^z \mathcal{T}_-(-u - 2\eta)$$

satisfy (9). The proof proceeds by substituting into (9), and making use of the properties of the R-matrix:

$$R_{21}^{st_1 \mathrm{i}st_2}(-u_1 + u_2) = R_{12}(-u_1 + u_2)\sigma^z \otimes \sigma^z$$

$$R_{12}^{st_1 \mathrm{i}st_2}(-u_1 + u_2) = R_{21}(-u_1 + u_2)\sigma^z \otimes \sigma^z$$

$$R_{12}(-u_1 - u_2 - 4\eta) = \sigma^z \otimes 1 R_{21}(-u_1 - u_2 - 4\eta) 1 \otimes \sigma^z$$

$$R_{21}(-u_1 - u_2 - 4\eta) = \sigma^z \otimes 1 R_{12}(-u_1 - u_2 - 4\eta) 1 \otimes \sigma^z \,.$$

Therefore the supermatrix $K_+(u)$ takes the form

$$K_+(u) = \begin{pmatrix} \sin(u+2\eta-\xi_+) & -\frac{\alpha_+ \sin\xi_+ \sin 2(u+2\eta)}{\mathrm{i}\sin 2\eta} \\ -\frac{\beta_+ \sin\xi_+ \sin 2(u+2\eta)}{\mathrm{i}\sin 2\eta} & \sin(u+2\eta+\xi_+) \end{pmatrix}. \quad (17)$$

Evidently, the results are consistent with those obtained from the Lax pair construction [18].

Now we can show that the Hamiltonian (1) is related to the transfer matrix $\tau(u)$:

$$\tau(u) = \tau(0)[1 - \sin 2\eta(H + \text{constant})u + \cdots] . \quad (18)$$

Thus we have shown that the model under study admits an infinite number of conserved currents which are in involution with each other.

In conclusion, we have presented the graded version of Sklyanin's formalism for the 1D small-polaron open chain. The boundary K-supermatrices thus constructed are consistent with those obtained from the Lax pair formulation [18]. Thus our construction shows that there is a graded version of Sklyanin's formalism to describe integrable systems with both boson and fermion fields in the finite interval. The extension of our construction to other open fermion chains, such as the supersymmetric t–J open chain [19], the 1D Hubbard open chain [7] and the 1D Bariev open chain [8], is also interesting and will be considered elsewhere.

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Note added in proof. After submitting this work, the author was informed that the graded reflection equation was first considered by L Mezincescu and R I Nepomechie in [20]. I thank one of the referees for drawing our attention to this reference.

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